Approved by Undergraduate Council

Sharon Gill

Digitally signed by Sharon Gill
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Date: 2011.01.13 13:54:43 -05'00'

General Education Course Approval Form

Date of Submission: **April 22, 2010**

1. Check which area(s) this course applies to.

Inquiry – Arts & Creativity	☐	Composition & Communications - II	☐
Inquiry – Humanities	☐	Quant Reasoning – Math	x
Inquiry – Nat/Math/Phys Sci	☐	Quant Reasoning – Stat	☐
Inquiry – Social Sciences	☐	Citizenship – USA	☐
Composition & Communications - I	☐	Citizenship - Global	☐

2. Provide Course and Department Information.

Department: Mathematics

Course Prefix and Number: MA 137   Credit hours: 4

Course Title: Calculus I with Life Science Applications

Expected Number of Students per Section: 25   Course Required for Majors in your Program? no

Prerequisite(s) for Course? *Math ACT of 27 or above, or math SAT of 620 or above, or MA 109 and MA 112, or MA 110, or consent of the department.*

Departmental Contact Information   Date:

Name: Zhongwei Shen   Email: zshen2@email.uky.edu

Office Address: POT 721   Phone: 257-3470

3. In addition to this form, the following must be submitted for consideration:

- A major course change form for revision of existing courses or a new course form for new courses.
- A syllabus that conforms to the Senate Syllabi Guidelines, including listing of the Course Template Student Learning Outcomes.
- A narrative that explains: 1) how the course will address the General Education and Course Template Learning outcomes; 2) active learning activities for students; and 3) the course assignment(s) that can be used for Gen Ed course assessment.

4. Signatures

Department Chair: [signature]   Date: April 22, 2010

Dean: Anna R. K. Bosch [signature]   Date: 4/23/10

Submit all proposals electronically to:
Sharon Gill
Office of Undergraduate Education
Sharon.Gill@uky.edu

Revised March 2010

To: Arny Stromberg, chair quantitative foundations vetting committee
From: Russell Brown, dus math
Subject: Proposal to include MA 137, Calculus I with Life Science Applications in the new General Education program
Date: April 12, 2010

Attached, find a syllabus and sample assignments for MA 137, Calculus I with Life Science Applications. This is a new course that was recently approved by the College of Arts and Sciences Educational Policy Committee and is now wending its way through the bureaucracy. We request that this course be approved as one path that students may use to satisfy part III a), quantitative foundations, of the proposed new General Education program. We expect that this course will be taken by students who are considering a course of study with high expectations for quantitative skills. The department intends to offer a variety of courses to allow students to choose a course that is appropriate to their intended course of study.

Learning outcome 1. Students in MA 137, Calculus I, are asked to write functional models and then use Calculus to answer questions about these models. Typical problems include writing a quadratic function to give the height of an object that is falling under the influence of gravity One might might be asked to find the height a particular time or to find the time when the object is at its highest point. The emphasis is not on memorizing formulae which answer questions, but being able to lay out the reasoning behind these formulae.

Another class of problems arise from geometry where one is asked to find tangent lines to curves with specified properties. Again, these problems have concrete realizations which can be visualized by drawing graphs on paper or with the aid of prosthetic devices such as a calculator. In these problems, students are expected to relate algebraic or symbolic information to geometric properties of the graphs.

The example assignments include two applications to biology. One tries to explain observations about arterial branching and the other is simple population model.

We believe that this course provides ample preparation in studying deterministic relationships between numerical quantities and will prepare students for the more sophisticated statistical inferential reasoning where relationships are studied with an additional element of uncertainty added.

Learning outcome 2. Students in MA 137 will be expected, at least at a rudimentary level, to provide correct explanations for the mathematical statements that they make. In recitation classes, students are given the opportunity to make presentations to their peers, on exams students are expected to provide written justification of their work, and students are given a small number of longer written assignments that ask them to work through a several step solution to a mathematical or applied problem and provide written justification of their work. Web homework is designed to include a problems which approach an idea from a variety of directions, rather than series of rote exercises. We do not typically ask for in depth explanations on web homework problems as we are not yet ready to automatically grade these responses. Rather, we try to develop good habits by providing students with a variety of problems so that memorizing recipes is not an efficient way of producing correct solutions.

In addition to providing examples of correct reasoning through presenting mathematics, students are asked to study examples that illustrate typical logical fallacies. Thus, a student will see the distinction between a logical implication (if a function is differentiable, then it is continuous) its converse (if a function continuous, then it is differentiaable) by studying examples of functions which are continuous, but not differentiable.

Information literacy. We believe that mathematics is a fundamental skill in processing information and thus the information literacy component of the course will be focused primarily on the deductive reasoning techniques that we are working to develop in our students.

In addition, we present a small number of additional assignments that ask students to find more information about the historical development of the subject of Calculus.

To assess our students learning, we will rely on the longer written assignments. For the past few semesters, students have been asked to complete six written assignments and these are graded by teaching assistants using a uniform rubric developed by the course coordinator. The ideal assignment will ask a student to formulate a more-or-less real life problem in mathematical terms, illustrate this problem by examples, develop a careful mathematical solution of this problem, and interpret the mathematics in terms of the original problem. Examples of such problems from are attached to this proposal. Several of these assignments are locally written (though we are not so bold as to claim that we have invented a new Calculus problem) and others are taken from the textbook that we use for MA 113, Calculus (Early Transcendentals), 6th edition by James Stewart.

Attached to this memorandum, please find a syllabus and course calendar for MA 137 and sample written assignments. Several of these have been used in past semesters in MA 113, several are new and involve biological applications. A few are intended to address the information literacy component of the new General Education program and have not been used in the past.

Attachments:

- Proposed syllabus for MA 137.
- Sample written assignment: Fish Management
- Sample written assignment: Arterial branching
- Project from Stewart: Early methods for finding tangents
- Project from Stewart: Newton, Leibniz, and the invention of Calculus

MA 137 001 Calculus I with life science applications

Time: MWF 9-9:50, lecture, TR 9:30-10:20 recitation.
Instructor: Alberto Corso
Office: 701 Patterson Office Tower
Mailbox: 715 Patterson Office Tower
Phone: 859 257-3167 (or 859 257-3336 to leave a message)
Email: corso@ms.uky.edu
Office Hours: TR 11:00–12:15, and by appointment.

Course overview: In Calculus I with life science applications, we will learn about derivatives, integrals and the fundamental theorems of calculus. We begin by introducing the notion of a limit. Limits are essential to defining derivatives and integrals. By the end of the semester students should know precise definitions of the derivative and the integral and the fundamental theorem of calculus which gives the relation between the derivative and the integral. We will illustrate the methods and ideas of calculus by studying several problems from biology. We will learn the interpretation of the derivative as a rate of change, and model growth and declines of populations.

Student learning outcomes:

Students will compute fluently.

Students will apply the methods of calculus in new contexts to solve unfamiliar problems.

Students will write correct justifications for their solutions to problems.

Course outline

Preview and review
    Preliminaries, elementary Functions, and graphing

Discrete time models, sequences, and difference equations

    Exponential growth and decay
    Sequences
    More population models

Limits and continuity

    Limits
    Continuity
    Limits at infinity
    The Sandwich Theorem and some trigonometric limits
    Properties of continuous functions

Differentiation

    Formal definition of the derivative
    The power rule, the basic rules of differentiation, and the derivatives of polynomials

The product and quotient rules, and the derivatives of rational and power functions
The chain rule and higher derivatives
Derivatives of trigonometric functions
Derivatives of exponential functions, exponential growth of populations
Derivatives of inverse and logarithmic functions
Approximations and local linearity

Applications of differentiation

Extrema and the Mean Value Theorem
Monotonicity and Concavity
Extrema, inflection points, and graphing
Optimization
L'Hospital's rule
Difference equations: stability, logistic growth

Integration

The definite integral
The Fundamental Theorem of Calculus
Applications of integration

Text: *Calculus for Biology and Medicine* by Claudia Neuhauser.

Class Attendance and Participation: This class is designed for active involvement of the students. You will be actively supporting each other as you gain experience and understanding. Multiple ideas and points of view are important. You will benefit from hearing others' approaches to problem solving, and they will benefit from you. So attendance and active participation are expected and contribute toward your grade.

Homework: There will be regular homework assignments. Weekly quizzes will be given that are taken from the homework.

Examinations: There will be three examinations and a final.

Grading:

Class attendance and participation	10%
Homework quizzes	10%
Exams	60%
Final	20%

Grading scale:

Lowest A	90%.
Lowest B	80%
Lowest C	70%
Lowest D	60%
E	Below 60%

Working Together: Students are encouraged to work together on homework, however, they must be sure to master the material from their collaborative work. It would be best for your own understanding if you put aside your notes from the discussions with your classmates and wrote up the solutions entirely from scratch. Working together on exams, of course, is expressly forbidden.

Absences: See Students Rights and Responsibilities,
www.uky.edu/StudentAffairs/Code/part2.html,
Section 5.4.2.2, for information about valid excused absences and their verification, and making up of missed assignments.

Cheating: Cheating and plagiarism can lead to significant penalties. See Sections 6.3 and 6.4 of Student Rights and Responsibilities,
www.uky.edu/StudentAffairs/Code/part2.html.

Expectations: I expect that everyone will maintain a classroom conducive to learning. I like an informal atmosphere, but it must be orderly. Thus, everyone is expected to behave with basic politeness, civility, and respect for others. In particular, talking in class is OK if it's part of a class discussion or directed to me. Private communications are not, especially during quizzes and tests. Neither are reading extraneous materials, using electronic equipment, or sleeping.

Accomodations for students with disabilities: If you have a documented disability that requires academic accommodations, please see me as soon as possible during scheduled office hours. In order to receive accommodations in this course, you must provide me with a Letter of Accommodation from the Disability Resource Center (Room 2, Alumni Gym, 257-2754, email address jkarnes@email.uky.edu)for coordination of campus disability services available to students with disabilities."

Suggestions: Suggestions for improvement are welcome at any time. Any concern about the course should be brought first to my attention. Further recourse is available through the offices of the Department Ombud and the Department Chair, both accessible from the Main Office in 715 Patterson Office Tower.

MA 137

Assignment 1 — Solutions

Iteration and Fish Management

Wildlife and fishery managers must establish appropriate harvesting plans in order to ensure proper control over the populations of animals and fish. This time we are going to look at several different strategies on harvesting fish for a population of fish in a controlled, landlocked lake. One of our basic assumptions will be that the natural growth rate of the population during a reproduction cycle depends upon the current size of that population. An equation for the population of fish $_nP$ in the nth reproduction cycle is given as

$$P_n = (1+r)P_{n-1} - \frac{r}{\ell}(P_{n-1})^2 - h_{n-1}$$

where

r	is the growth rate per cycle,
ℓ	is the capacity of fish in the lake, and
h_{n-1}	is the number of fish harvested during the previous cycle.

This is a recurrence relation, because the population at a later time depends on the population at an earlier time. P_0 is the initial population and P_1 is the population after the first cycle. It should be clear that P_1 will depend on the number of fish with which we start. This process is called *iteration.*

Let's suppose that for the type of fish and the size of the lake, prior research and experimentation has determined the capacity of fish in the lake to be $\ell=100,000$, and the growth rate of the population to be $r = 0.45$. Therefore the equation with which we will be working will be:

$$P_n = (1+0.45)P_{n-1} - \frac{0.45}{100,000}(P_{n-1})^2 - h_{n-1}$$

Suppose you stocked the new lake with 85,000 fish. So, we will start with this value: P_0=85,000.

Prospectus 1: What will happen if the fishery adopts the plan to harvest 20,000 fish during each reproduction cycle?

This means that we will take $h_{n-1} = 20,000$ for every value of n. The equation we need to iterate is

$$P_n = 1.45 \cdot P_{n-1} - 0.0000045 \cdot (P_{n-1})^2 - 20,000$$

We need to see what happens to the population of fish in the lake. If we harvest too many each season, then the number of fish will eventually dwindle down to nothing and we will be out of the fish business. If we do not harvest enough, then the lake will fill with too many fish, they will be under stress, they will get sick, no one will want to buy your fish; you get stressed, ... Let's just say that things won't go well.

Let's look at the population over the first 10 reproductive cycles. Thus, we want to know P_0, P_1, ..., P_{10}. To do this we would plug in P_0 to find P_1, then plug in P_1 to find P_2, *etc.*

$$P_1 = 1.45 \cdot P_0 - 0.0000045 \cdot (P_0)^2 - 20{,}000 = 70{,}738$$

$$P_2 = 1.45 \cdot P_1 - 0.0000045 \cdot (P_1)^2 - 20{,}000 = 60{,}052$$

Note two things: first, we truncated to the integer value, since a fractional value of a fish does not do the business any good. Second, I did not do these calculations by hand. I used a calculator. Can we make the calculator do all the work without have to punch in so many numbers?

Actually, we can handle iteration on a calculator or with a spreadsheet. On the TI-84 Plus, you can graph the values, or just getting values of the different iterations. First, we need to put the calculator in Sequence mode. Now, when you press the Y= key you get u(n)= and v(n)=. For our equation, we will enter

nMin = 0
u(n)=1.45*u(n-1) – .0000045(u(n-1))^2 – 20000
u(nMin)=85000

Now to graph the population over the first 10 reproductive cycles, press WINDOW. Fill in the following values and leave the others as you found them.

nMin =0	Xmax=10
nMax=10	Xscl =1
PlotStart=1	Ymin =0
PlotStep=1	Ymax=85000
Xmin =0	Yscl =10000

Then press GRAPH. Press the right arrow to trace the different iterates of P_n.

Problems:

1. Find P_3, ..., P_{10}. What happens? Will you stay in the fish business, or (pardon the obvious) is there something fishy going on here?

   Substituting n = 3, 4, ..., 10 into the formula for P_n, we get:

n	1	2	3	4	5	6	7	8	9	10
P_n	70738	60052	50848	42094	33063	23022	10997	-4598	-26763	-62029

The last three numbers do not make any sense in the framework of this problem, since you cannot produce a negative number of fish and negative fish (anti-fish, aunty-fish?) cannot reproduce themselves to get more negative fish. The table should look like the following if any thought has gone into the context of the problem.

n	1	2	3	4	5	6	7	8	9	10
P_n	70738	60052	50848	42094	33063	23022	10997	0	0	0

2. What will happen if you reduce the harvest to only 10,000 fish during each reproduction cycle?

   The formula changes to $P_n = 1.45 \cdot P_{n-1} - 0.0000045 \cdot (P_{n-1})^2 - 10{,}000$ and the numbers we get from the formula are:

n	1	2	3	4	5	6	7	8	9	10
P_n	80738	77736	75524	73824	72534	71499	70669	69997	69447	68995

   It looks as if the population is still tending to 0, but at a slower rate.

3. What will happen if you reduce the harvest even further? Try a harvest of 5,000.

   The formula changes to $P_n = 1.45 \cdot P_{n-1} - 0.0000045 \cdot (P_{n-1})^2 - 5{,}000$ and the numbers we get from the formula are:

n	1	2	3	4	5	6	7	8	9	10
P_n	85738	86240	86580	86809	86962	87064	87132	87178	87208	87228

   Here, the population of the fish is increasing.

Based on these two results you might think that there is a rate somewhere between 5,000 and 10,000 that will result in a constant population of 85,000. If we call this equilibrium rate y then we can find this stable rate by solving the equation

$$85{,}000 = 1.45 \cdot 85{,}000 - 0.0000045 \cdot 85{,}000^2 - y$$

for y .

4. How many fish can we harvest each time to keep a constant population of 85,000?

   We need to solve the above equation for y:

$$85{,}000 = 1.45 \cdot 85{,}000 - 0.0000045 \cdot 85{,}000^2 - y$$

$$y = 0.45 \times 85000 - 0.0000045 \cdot 85{,}000^2 = 5737.5$$

   What do we do with the $\frac{1}{2}$ fish? In other words, do we harvest 5737 fish per annum or 5738 fish per annum? What is the difference?

5. How many fish could we harvest if we were willing to keep a constant population of 75,000? 50,000?

   We need to set up and to solve the above equation for y when we have lim P_n=75000 and when lim P_n = 50000:

$$75{,}000 = 1.45\cdot 75{,}000 - 0.0000045\cdot 75{,}000^2 - y$$
$$y = 0.45\times 75000 - 0.0000045\cdot 75{,}000^2 = 8437.5$$

and

$$50{,}000 = 1.45\cdot 50{,}000 - 0.0000045\times 50{,}000^2 - y$$
$$y = 0.45\times 50{,}000 - 0.0000045\times 50{,}000^2 = 11{,}250$$

Bonus Problem: Prospectus 2

Consider some additional harvesting strategies. Instead of harvesting a fixed number of fish each cycle, suppose as fisheries manager you try harvesting a fixed percentage of the current population each cycle. Let's say you decide to harvest 20% of the fish in the lake each cycle. Then the equation becomes

$$P_n' = 1.45\cdot P_{n-1} - 0.0000045\cdot (P_{n-1})^2$$
$$h_n = 0.2P_n'$$
$$P_n = P_n' - h_n$$

You hope that this kind of proportional harvesting will help to stabilize the population, because when the population becomes smaller, fewer fish are harvested.

6. Is the population stabilizing? At what value? How could you tell?

   We will look at the limit, and, in fact, could look at the plot of the recursion on the calculator, or with Excel:

   $$P_n = 1.45\cdot P_{n-1} - 0.0000045\cdot (P_{n-1})^2 - 0.2\cdot P_{n-1}$$

   A table of values reveals that it is decreasing at a very slow rate. The limiting value, should it have one, would have to solve the equation:

   $$\lim_{n\to\infty} P_n = 1.45\cdot \lim_{n\to\infty} P_{n-1} - 0.0000045\cdot (\lim_{n\to\infty} P_{n-1})^2 - 0.2\cdot \lim_{n\to\infty} P_{n-1}$$
   $$P = 1.45\times P - 0.0000045\times P^2 - 0.2\times P$$
   $$0.0000045P^2 - 0.25P = 0$$
   $$0.0000045P\left(P - \frac{500000}{9}\right) = 0$$
   $$P = 55556$$

   This says that the population would stabilize at about 55,556.

7. What happens if you change the percentage of the population that you will harvest to 25%? Will the population stabilize for this business model?

   We will do the same as above, except this time our equation is:

   $$P_n = 1.45\cdot P_{n-1} - 0.0000045\cdot (P_{n-1})^2 - 0.25\cdot P_{n-1}$$

A table of values reveals that it is decreasing at a very slow rate. The limiting value, should it have one, would have to solve the equation:

$$\lim_{n\to\infty} P_n = 1.45 \cdot \lim_{n\to\infty} P_{n-1} - 0.0000045 \cdot (\lim_{n\to\infty} P_{n-1})^2 - 0.25 \cdot \lim_{n\to\infty} P_{n-1}$$

$$P = 1.45 \times P - 0.0000045 \times P^2 - 0.25 \times P$$

$$0.0000045P^2 - 0.2P = 0$$

$$0.0000045P\left(P - \frac{400000}{9}\right) = 0$$

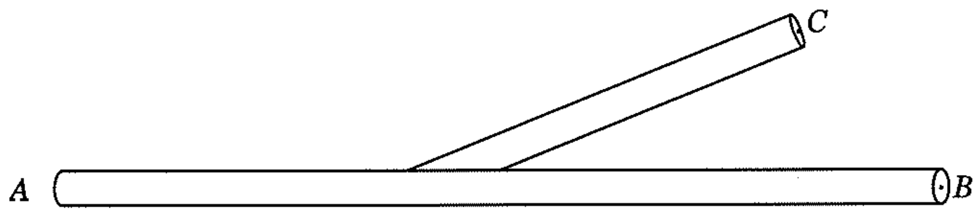
$$P = 44444$$

This says that the population would stabilize at about 44,444.

MA 137

Arterial Branching Calculus

In this time of quick diagnosis and many folks being taken to the hospital for heart and cardiopulmonary surgery, let us look at one problem that has arisen previously in optimizing the surgery that attaches a blood vessel to an artery. The surgeon must attach a small blood vessel to an existing artery AB in order to get a blood supply from point A to point C. In this surgical connection of the small artery to an existing larger artery, attention must be paid to minimizing the viscous resistance to the blood flow, that is the resistance to the blood flow for blood traveling from A to C.



The minimization of this viscous resistance will, in turn, minimize the strain on the heart, which is, after all, what the surgeon wants to do.

The flow of fluids through pipes has been well studied and, after all, blood is just a liquid flowing through a pipe – the artery. The fact that blood is more viscous than, say water, is important, but what is more important is that the closer the liquid is to the wall the more friction it has moving against the wall. In fact,

Jean Louis Marie Poiseuille was born in Paris, France, in 1797 and studied at the École Polytechnique in Paris. He was trained in mathematics and physics, but graduated in 1828 with a dissertation entitled *Recherches sur la force coeur aortique.* He was interested in the flow of human blood in narrow tubes. His work lead him to experimentally derive what is known as *Poiseuille's Law* (or the Hagen-Poiseuille Equation) that describes the voluminal laminar stationary flow[1] of an incompressible uniform viscous liquid through a cylindrical tube with constant cross-section. This seems to work reasonably well for small vessels and capillaries, though not for the larger arteries, such as the aorta, the brachial artery, or the femoral artery.

According to Poiseuille's Law resistance is proportional to the length of tube travelled and inversely proportional to the fourth power of the radius of the tube,

$$R = \frac{kd}{r^4} \tag{1}$$

where R is the resistance, k is a constant, d is the distance travelled along the tube, and r is the radius of the tube. The proportionality constant is determined by the viscosity of the blood. Also, blood vessels are not actually rigid, but for short distances, as is the case in surgery, than are nearly rigid.

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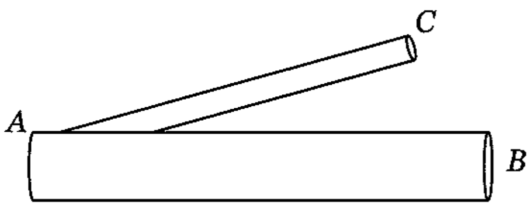
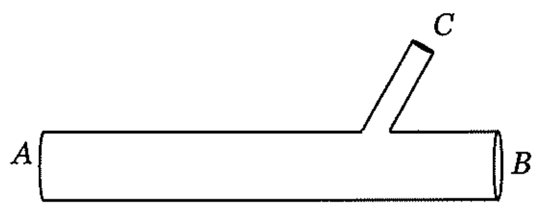
[1] *Laminar flow* means that all particles of the fluid pass through the tube along paths that are parallel to its wall, and that the rate of flow increases smoothly from the wall toward the center.

The surgeon's problem is where to make the attachment. If it is moved closer to *A* the path travelled by the blood is shortened, but the length travelled in the narrower tube is increased. If the join is closer to *B*, the path the blood travels is longer, but less of it is in the narrow tube. The problem, then, is to find the point between A and B at which to attach the vessel in order to minimize the resistance to the flow of blood.



Attachment closer to A

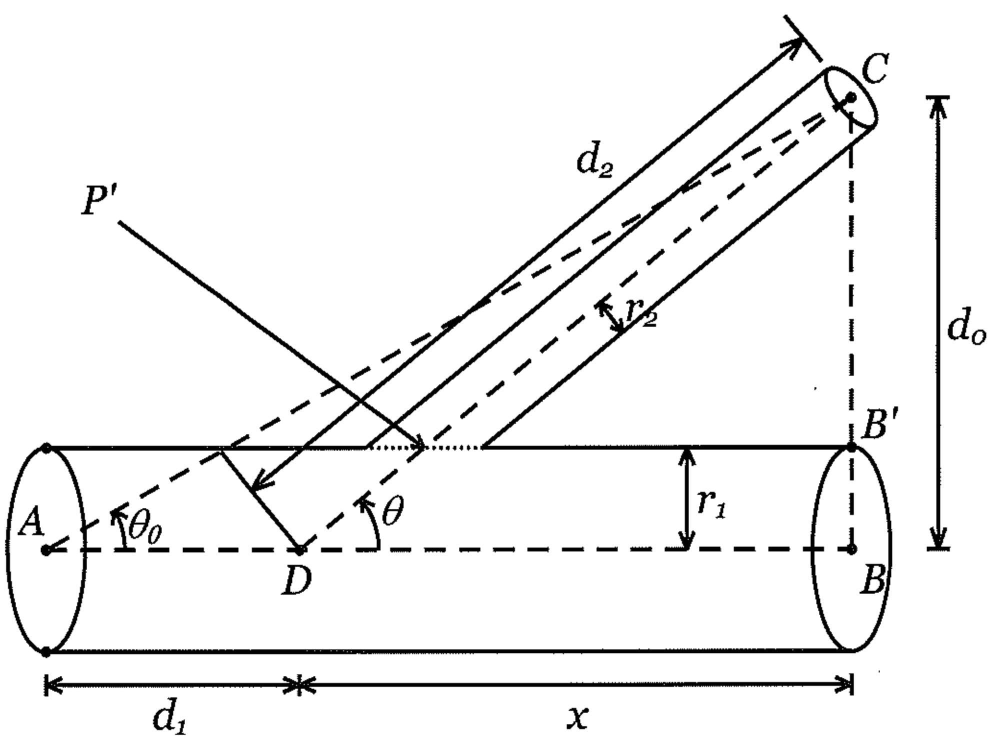


Attachment closer to B

It appears rather clear that as we move the point of attachment between *A* and *B*, we are changing the angle at which the smaller vessel is attached to the larger vessel, since the point *C* is fixed and cannot be moved.

The Resistance Equation:

We need to set up some notation for the problem. See the figure below:



Since the artery runs from *A* to *B*, we will be safe in choosing the point *B* so that the angle $\angle ABC$ is a right angle. Of the above quantities we know:

d_0 – the distance from *B* to *C*,
r_1 – the radius of the artery into which the branch is being placed,

r_2 – the radius of the branch that is being included,
θ_0 – the angle $\angle BAC$, and
$d_1 + x$ – the length of the artery onto which the branch is to be placed.

The variables are

θ – the branching angle ,
d_1 – the distance from A to the center line of the branch,
d_2 – the length of the branch,
x – the distance from the center line of the branch to the point B.

Our problem is to find the value of the angle θ that minimizes the total resistance to the flow of blood from A to C.

First in order to be able to make use of one variable calculus, we need to express the other variables in terms of θ. From our diagram above we see that:

$$\csc\theta = \frac{d_2}{d_0}$$

$$\cot\theta_0 = \frac{d_1 + x}{d_0}, \text{ and}$$

$$\cot\theta = \frac{x}{d_0}$$

From this it follows that

$$\begin{aligned} d_2 &= d_0 \csc\theta, \\ d_1 &= d_0 \cot\theta_0 - x, \\ x &= d_0 \cot\theta, \text{ and thus} \\ \mathrm{d}_1 &= d_0 (\cot\theta_0 - \cot\theta) \end{aligned}$$

Now, if $R = R(\theta)$ is the total resistance to the flow of blood, then R is the sum of the resistance through the artery plus the resistance through the branch vessel. From Equation (1) we have:

$$\begin{aligned} R_{TOTAL} &= R_{artery} + R_{branch} \\ R &= \frac{kd_1}{r_1^4} + \frac{kd_2}{r_2^4} \\ &= k\left[\frac{d_0 \cot\theta_0 - d_0 \cot\theta}{r_1^4} + \frac{d_0 \csc\theta}{r_2^4}\right] \end{aligned}$$

Since k, d_0, r_1 and θ_0 are constants, we can simplify the above equation by putting

$$K = \frac{kd_0}{r_1^4} \cot\theta_0$$

which is a constant. This simplifies the above equation to:

$$R(\theta) = K + kd_0\left[\frac{\csc\theta}{r_2^4} - \frac{\cot\theta}{r_1^4}\right] \tag{2}$$

Minimizing the Resistance

Without any loss of generality, we may assume that $0 < \theta < \pi$. Now, we know how to proceed. We need to differentiate the resistance equation with respect to θ. From our equation (2) above, we find that:

$$R'(\theta) = kd_0\left[\frac{-\csc\theta\cot\theta}{r_2^4} - \frac{\csc^2\theta}{r_1^4}\right] \tag{3}$$

Now, we have to set $R'(\theta) = 0$ and solve for θ.

$$kd_0\left[\frac{-\csc\theta\cot\theta}{r_2^4} + \frac{\csc^2\theta}{r_1^4}\right] = 0$$

$$\frac{\csc\theta\cot\theta}{r_2^4} - \frac{\csc^2\theta}{r_1^4} = 0$$

$$\csc\theta\left(r_1^4\cot\theta - r_2^4\csc\theta\right) = 0$$

$$r_1^4\cot\theta - r_2^4\csc\theta = 0$$

$$r_1^4\frac{\cos\theta}{\sin\theta} - r_2^4\frac{1}{\sin\theta} = 0$$

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

$$\theta = \arccos\left(\frac{r_2^4}{r_1^4}\right) \tag{4}$$

Problem 1: Verify each of the steps in the solution above.

Problem 2: Verify that the value of θ found above yields a minimum for θ.

Note that the critical value of θ depends only on the radii of the tubes. This tells us that the locations of the points A, B, and C are not crucial to the work. The only real place where they have an effect is in the assumption that the lengths were such that we are able to assume that the artery and vessel were rigid.

Here is an important point. In this problem the surgeon is concerned with the location of that the resistance is minimized. He is not interested in the actual amount of the resistance. We do not need to find R_{min} here, then.

Final Remarks:

It is interesting to note that experimental observations have shown that the angle given in Equation (4) is close to the actual angles at which blood vessels are attached to arteries in the body.

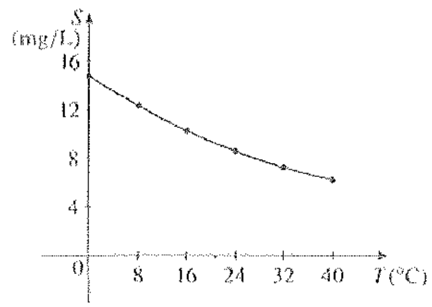
Now, the surgeon is still waiting – and, of course, the patient would like to have the operation in order to relieve heart stress. If you report the value of θ that we found above, he or she will still not know where to attach the vessel. You must help.

Problems:

1. Verify each of the steps in the solution above.
2. Verify that the value of θ found above yields a minimum for θ.
3. Find $R''(\theta)$.
4. With reference to the second figure above, for a known value of θ (such as the value we found in Equation (4), find the distance $P'B'$ in terms of θ, d_0 and r_1.

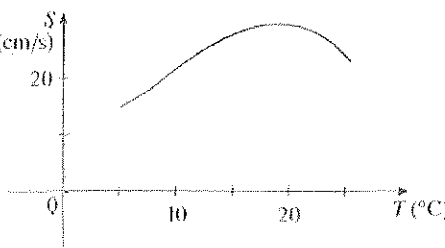
48. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

49. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the value of $S'(16)$ and interpret it.



Adapted from *Environmental Science: Science Lives Within the System of Nature*, 2d ed., by Charles E. Kupchella, © 1989. Reprinted by permission of Prentice-Hall, Inc., Upper Saddle River, NJ

50. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



51–52 Determine whether $f'(0)$ exists.

51. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

52. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

WRITING PROJECT

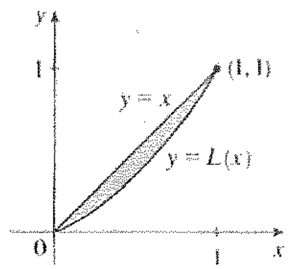
EARLY METHODS FOR FINDING TANGENTS

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen further than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's teacher at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

The following references contain explanations of these methods. Read one or more of the references and write a report comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point $(1, 3)$ and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1989). pp. 389, 432.
2. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.
3. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
4. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

households receive $a\%$ of the income, in which case the Lorenz curve would be the line $y = x$. The area between the Lorenz curve and the line $y = x$ measures how much the income distribution differs from absolute equality. The *coefficient of inequality* is the ratio of the area between the Lorenz curve and the line $y = x$ to the area under $y = x$.



(a) Show that the coefficient of inequality is twice the area between the Lorenz curve and the line $y = x$, that is, show that

$$\text{coefficient of inequality} = 2\int_0^1 [x - L(x)]\,dx$$

(b) The income distribution for a certain country is represented by the Lorenz curve defined by the equation

$$L(x) = \tfrac{5}{12}x^2 + \tfrac{7}{12}x$$

What is the percentage of total income received by the bottom 50% of the households? Find the coefficient of inequality.

68. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

(a) Use a graphing calculator or computer to model these data by a third-degree polynomial.

(b) Use the model in part (a) to estimate the height reached by the *Endeavour*, 125 seconds after liftoff.

WRITING PROJECT

NEWTON, LEIBNIZ, AND THE INVENTION OF CALCULUS

We sometimes read that the inventors of calculus were Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). But we know that the basic ideas behind integration were investigated 2500 years ago by ancient Greeks such as Eudoxus and Archimedes, and methods for finding tangents were pioneered by Pierre Fermat (1601–1665), Isaac Barrow (1630–1677), and others. Barrow—who taught at Cambridge and was a major influence on Newton—was the first to understand the inverse relationship between differentiation and integration. What Newton and Leibniz did was to use this relationship, in the form of the Fundamental Theorem of Calculus, in order to develop calculus into a systematic mathematical discipline. It is in this sense that Newton and Leibniz are credited with the invention of calculus.

Read about the contributions of these men in one or more of the given references and write a report on one of the following three topics. You can include biographical details, but the main thrust of your report should be a description, in some detail, of their methods and notations. In particular, you should consult one of the sourcebooks, which give excerpts from the original publications of Newton and Leibniz, translated from Latin to English.

- The Role of Newton in the Development of Calculus
- The Role of Leibniz in the Development of Calculus
- The Controversy between the Followers of Newton and Leibniz over Priority in the Invention of Calculus

References

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1987), Chapter 19.

2. Carl Boyer, *The History of the Calculus and Its Conceptual Development* (New York: Dover, 1959), Chapter V.
3. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), Chapters 8 and 9.
4. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), Chapter 11.
5. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Leibniz by Joseph Hofmann in Volume VIII and the article on Newton by I. B. Cohen in Volume X.
6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), Chapter 12.
7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), Chapter 17.

Sourcebooks

1. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987), Chapters 12 and 13.
2. D. E. Smith, ed., *A Sourcebook in Mathematics* (New York: Dover, 1959), Chapter V.
3. D. J. Struik, ed., *A Sourcebook in Mathematics, 1200–1800* (Princeton, N.J.: Princeton University Press, 1969), Chapter V.

5.5 THE SUBSTITUTION RULE

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1 + x^2}\, dx$$

To find this integral we use the problem-solving strategy of *introducing something extra*. Here the "something extra" is a new variable; we change from the variable x to a new variable u. Suppose that we let u be the quantity under the root sign in (1), $u = 1 + x^2$. Then the differential of u is $du = 2x\,dx$. Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in (1) and so, formally, without justifying our calculation, we could write

▪ Differentials were defined in Section 3.10. If $u = f(x)$, then

$$du = f'(x)\,dx$$

$$\boxed{2} \qquad \int 2x\sqrt{1 + x^2}\, dx = \int \sqrt{1 + x^2}\, 2x\, dx = \int \sqrt{u}\, du$$
$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2 + 1)^{3/2} + C$$

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(x^2 + 1)^{3/2} + C\right] = \tfrac{2}{3} \cdot \tfrac{3}{2}(x^2 + 1)^{1/2} \cdot 2x = 2x\sqrt{x^2 + 1}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))g'(x)\,dx$. Observe that if $F' = f$, then

$$\boxed{3} \qquad \int F'(g(x))g'(x)\,dx = F(g(x)) + C$$